TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 320, Number 2, August 1990

ENDOMORPHISM RINGS OF FORMAL A_0-MODULES

SHUJI YAMAGATA

ABSTRACT. Let A_0 be the valuation ring of a finite extension K_0 of Q_p and $A \supset A_0$ be a complete discrete valuation ring with the perfect residue field. We consider the endomorphism rings of n-dimensional formal A_0-modules Γ over A of finite A_0-height with reduction absolutely simple up to isogeny. Especially we prove commutativity of $\mathrm{End}_{A,A_0}(\Gamma)$. Given an arbitrary finite unramified extension K_1 of K_0, a variety of examples (different dimensions and different A_0-heights) is constructed whose absolute endomorphism rings are isomorphic to the valuation ring of K_1.

Let K_0 be a finite extension of Q_p and A_0 the valuation ring of K_0; let $K \supset K_0$ be a complete discrete valuation field with the perfect residue field k of characteristic $p > 0$; let A be the valuation ring of K.

It is known that the fraction field of the endomorphism ring of a one-dimensional formal group of height h over A is a finite extension of Q_p of degree dividing h (cf. Lubin [7]).

In Theorem 1 and Proposition 2, we prove a higher-dimensional analogue of the above fact: if an n-dimensional formal A_0-module Γ over A satisfies an assumption that the reduction $\Gamma_k = \Gamma \otimes_A k$ of Γ is an absolutely simple formal A_0-module up to isogeny and of finite A_0-height $h \geq n$ (h is relatively prime to n), then we prove that the fraction field Λ of the endomorphism ring of Γ over A as a formal A_0-module is a finite extension field of K_0 of degree dividing h such that $e(\Lambda/K_0)$ divides $e(K/K_0)$ and that $f(\Lambda/K_0)$ divides $f(K/K_0)$ if $f(K/K_0)$ is finite.

In the corollary of Theorem 2, we give examples: for any positive integers h and n with $h \geq n+1$ ($h \geq 1$ if $n = 1$) and for any positive divisor g of h, there exists an n-dimensional formal A_0-module over A_0 of A_0-height h whose absolute A_0-endomorphism ring is the valuation ring of the unramified extension of K_0 of degree g (cf. Cox [1] and Yamasaki [11]).

Acknowledgment. The author would like to thank Professor Tsuneo Kanno and Professor Tetsuo Nakamura for their valuable advice and constant encouragement. The author would also like to thank the referee for his kind and helpful suggestions to improve on the original version.

Received by the editors February 22, 1988 and, in revised form, October 10, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14L05.
Key words and phrases. Formal modules (groups), endomorphism rings.

1. Notations

In this paper, a field means a commutative field and we use the following notations.

$p =$ a prime number.

$Z_p =$ the ring of p-adic integers.

$Q_p =$ the field of p-adic numbers.

$K_0 =$ a finite extension of Q_p with the residue field k_0.

$A_0 =$ the valuation ring of K_0.

K $(\supset K_0) =$ a complete discrete valuation field with the perfect residue field k of characteristic $p > 0$.

$\pi =$ a prime element of K.

$A =$ the valuation ring of K.

$\bar{k} =$ an algebraic closure of k.

$[M : C] =$ the dimension of a vector space M over a field C.

For an extension E/E' of complete discrete valuation fields,

$e(E/E') =$ the ramification index of E over E'.

$f(E/E') =$ the residue degree of E over E'.

$M_t(T) =$ the total matrix ring of degree t over a ring T.

$(a, b) =$ the (positive) greatest common divisor of integers a and b.

2. Endomorphism rings

In this paper, a formal group means a formal group law.

An n-dimensional formal A_0-module over a commutative A_0-algebra S is an n-dimensional commutative formal group over S such that there is an endomorphism $[a]$ of a formal group over S for each $a \in A_0$ whose Jacobian matrix is aI_n ($I_n =$ the unit matrix of degree n) and that $a \to [a]$ is a ring homomophism. If a formal A_0-module is of height H as a formal group, we define the A_0-height of Γ as the number $H/[K_0 : Q_p]$ (cf. [2], [3, III. 4.3, 5.5] and [5, V. 29.7.2]). A formal A_0-homomorphism over S between formal A_0-modules is a homomorphism over S of the formal groups which commutes with $[a]$ for all $a \in A_0$. We write $\mathrm{End}_{S, A_0}(\Psi)$ the formal A_0-endomorphism ring of a formal A_0-module Ψ over S. An isogeny over S between formal A_0-modules is a formal A_0-homomophism that, as a homomorphism of formal groups, is an isogeny.

Let Γ be an n-dimensional formal A_0-module over A of finite A_0-height. Let $\Gamma_k = \Gamma \otimes_A k$ be the formal A_0-module over k obtained by reducing the coefficients of Γ modulo the maximal ideal of A.

We put $\Lambda = Q_p \otimes_{Z_p} \mathrm{End}_{A, A_0}(\Gamma)$. Λ is a K_0-algebra. As Γ is of finite A_0-height, we identify Λ with its image in $Q_p \otimes_{Z_p} \mathrm{End}_{k, A_0}(\Gamma_k)$ through reduction (cf. [5, IV.21.8.19]).

Let $\mathrm{END}_{*, A_0}(\Gamma)$, the absolute A_0-endomorphism ring of Γ, be the union of $\mathrm{End}_{B, A_0}(\Gamma \otimes_A B)$ where B runs over all the valuation rings of finite extensions of K. ($\Gamma \otimes_A B$ is the scalar extension of Γ to B.)

We assume,

(∗) Γ_k is an n-dimensional formal A_0-module such that the scalar extension $\Gamma_{\overline{k}} = \Gamma_k \otimes_k \overline{k}$ of Γ_k to $\overline{k}$ is simple as a formal A_0-module up to isogeny (i.e. Γ_k is absolutely simple up to isogeny) and of finite A_0-height h ($\geq n$).

Remark 1. (i) $(h, n) = 1$ by [5, V.29.8.3] (cf. [3, III.4, Corollary 2 of Proposition 8]).

(ii) For examples of Γ and Γ_k satisfying (∗), see §4 and [9, Proposition 5].

We put $D = Q_p \otimes_{Z_p} \mathrm{End}_{\overline{k}, A_0}(\Gamma_{\overline{k}})$. By assumption (∗), D is a division algebra over K_0. We put $\Omega = \Lambda \otimes_{K_0} K$.

The following simple proof of Proposition 1 is due to the referee.

Proposition 1. *Under our assumption* (∗), D *is a central division algebra over* K_0 *of dimension* h^2.

Proof. By [8, II] (or [5, V.28.5.9]), $\Gamma_{\overline{k}}$ is isogeneous to a product $\Gamma_1^{t_1} \times \Gamma_2^{t_2} \times \cdots \times \Gamma_m^{t_m}$, where each Γ_i is simple up to isogeny of finite height H_i, Γ_i is not isogeneous to Γ_j for $i \neq j$, and the decomposition is unique up to isogeny. All this is taking place in the category of formal groups, not of formal A_0-modules. Δ denotes the scalar extension $Q_p \otimes_{Z_p} \mathrm{End}_{\overline{k}, Z_p}(\Gamma_{\overline{k}})$ of the endomorphism ring of $\Gamma_{\overline{k}}$ as a formal group. It follows that $\Delta \cong \bigoplus M_{t_i}(\Delta_i)$, where each Δ_i is a central division algebra over Q_p of dimension H_i^2. From the definition of Γ as a formal A_0-module, we see that K_0 injects into Δ and that D is the commutant of K_0 in Δ. Since D is a division algebra, the center of Δ must be a field. Thus Γ is isogeneous to $\Gamma_1^{t_i}$ and $\Delta \cong M_{t_1}(\Delta_1)$. From standard theorems about central division algebras, we see that D is a central division algebra over K_0 (double commutant theorem) and that

$$[D : Q_p][K_0 : Q_p] = [\Delta : Q_p] = t_1^2 H_1^2 .$$

Thus we have

$$[D : K_0] = \left(\frac{t_1 H_1}{[K_0 : Q_p]} \right)^2 = h^2 ,$$

where the last equality follows from the definition of A_0-height.

Let L be the tangent space (or the Lie algebra) of the scalar extension $\Gamma \otimes_A K$ of Γ to K. Then L is an n-dimensional vector space over K, a faithful Λ-module and a bimodule over Λ and K. L is thus a nontrivial module over Ω (cf. [5, II.14.2]).

Theorem 1 is a higher-dimensional analogue of [7, Theorem 2.3.2] (or [5, IV. 23.2.6]).

Theorem 1. *Under our assumption* $(*)$, Λ *is a finite extension field of* K_0.

Proof. By Proposition 1, the K_0-subalgebra Λ of D is a division algebra over K_0 of dimension dividing h^2. Let Z be the center of Λ. Then we have $[\Lambda : Z] = h'^2$ with h' dividing h. $Z \otimes_{K_0} K$ is a finite direct sum of finite extensions K_i $(\supset Z)$ of K. Hence every minimal left ideal of $\Lambda \otimes_Z K_i$ has dimension over K divisible by h', and so does every minimal ideal of $\Omega \cong \Lambda \otimes_Z (Z \otimes_{K_0} K)$. Ω is a semisimple K-algebra. Since L is a nontrivial Ω-module of dimension n over K, h' divides n.

On the other hand, $(h, n) = 1$ and h' divides h. Thus $h' = 1$ and Λ is a field.

Remark 2. If Γ_k is absolutely simple up to isogeny and of A_0-height ∞, then we have $\dim \Gamma = \dim \Gamma_k = \dim \Gamma_{\bar{k}} = 1$ (cf. [5, V.29.8.3]) and so $\mathrm{End}_{A, A_0}(\Gamma)$ is commutative.

Proposition 2. *Under our assumption* $(*)$, $[\Lambda : K_0]$ *divides* h. *Furthermore* $e(\Lambda/K_0)$ *divides* $e(K/K_0)$ *and* $f(\Lambda/K_0)$ *divides* $f(K/K_0)$ *if* $f(K/K_0)$ *is finite.*

Proof. By Theorem 1, Λ $(\supset K_0)$ is a subfield of D. Therefore, by Proposition 1, $[\Lambda : K_0]$ divides h.

Let F be a minimal ideal of Ω. By Theorem 1, F is a composite of K and Λ over K_0. We have

$$e(F/\Lambda)e(\Lambda/K_0) = e(F/K)e(K/K_0).$$

Then $e(\Lambda/K_0)/(e(\Lambda/K_0), e(K/K_0))$ divides $e(F/K)$ and so $[F : K]$. Therefore $e(\Lambda/K_0)/(e(\Lambda/K_0), e(K/K_0))$ divides $n = [L : K]$, since Ω is semisimple and L is a nontrivial Ω-module.

On the other hand, $e(\Lambda/K_0)$ divides $[\Lambda : K_0]$ and so h. Hence, by $(h, n) = 1$, $e(\Lambda/K_0)$ divides $e(K/K_0)$.

For the residue degrees, the same arguement holds if $f(K/K_0)$ is finite.

Corollary (of Theorem 1 and Proposition 2). *Under our assumption* $(*)$, *the absolute* A_0*-endomorphism ring of* Γ *is commutative. Its fraction field is an extension of* K_0 *of degree dividing* h *and has the ramification index dividing* $e(K/K_0)$.

Proof. Let K^* be the composite of K and the fraction field of the Witt vector ring over $\bar{k}$. Let A^* be the valuation ring of K^*. We remark $e(K_*/K_0) = e(K/K_0)$. By [10, Theorem 3.2] (or [5, IV.23.2.2]) and [5, IV.21.1.4, Remarks (ii)], $\mathrm{END}_{*, A_0}(\Gamma)$ is contained in $\mathrm{End}_{A^*, A_0}(\Gamma \otimes_A A^*)$. Hence our result follows from Theorem 1 and Proposition 2.

3. A LEMMA

Let K' $(\subset K)$ be the composite of K_0 and the fraction field (in K) of the Witt vector ring over k. Then K is a totally ramified finite extension of K' and

$e(K'/K_0) = 1$. Let A' be the valuation ring of K'. Let τ' be the Frobenius of K' over K_0 (i.e. the K_0-automorphism of K' satisfying $a^{\tau'} \equiv a^{p^{f(K_0/Q_p)}}$ (mod the maximal ideal of A') for all $a \in A'$).

In §§3 and 4, we assume that there exists an extension τ of τ' to an automorphism of K.

We write R the A-module of formal power series

$$x = a_h t^h + a_{h+1} t^{h+1} + \cdots + a_n t^n + \cdots$$

in an indeterminant t, with coefficients $a_n \in A$, where the exponent h is arbitrary. The A-module R is made into a ring by the multiplication law

$$(at^i)(bt^j) = (ab^{\tau^i})t^{i+j} \quad \text{for all } a, b \in A.$$

We also write $A_\tau[[t]]$ the subring of R with only terms of nonnegative exponents (cf. Hilbert-Witt ring and localized Hilbert-Witt ring in [3, III.4.1]).

Lemma (a generalization of the claim in [3, III.5, Proof of Theorem 3]). *Let $x = \sum a_i t^i \in A_\tau[[t]]$ with $a_i \in A$ be such that $v(a_i) > 0$ for all $0 \le i \le s-1$ and $v(a_s) = 0$, where v is the normalized discrete valuation of K with $v(\pi) = 1$. Suppose that $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to the left $A_\tau[[t]]$-ideal $A_\tau[[t]]x$ generated by x. Then we have $u = 0$, i.e. $b_i = 0$ for all $0 \le i \le s-1$.*

Proof. We remark that v is invariant under τ. We take $\sum c_i t^i \in A_\tau[[t]]$ such that

$$\sum b_h t^h = \left(\sum c_i t^i\right)\left(\sum a_j t^j\right) = \sum\sum c_i a_j^{\tau^i} t^{i+j}.$$

Then we have, for all integers $h \ge 0$,

$$0 = c_{s+h} a_0^{\tau^{s+h}} + c_{s+h-1} a_1^{\tau^{s+h-1}} + \cdots + c_0 a_{s+h}.$$

Hence we have, for all integers $h \ge 1$,

$$\begin{aligned} v(c_h) &= v(c_h) + v(a_s^{\tau^h}) = v(c_h a_s^{\tau^h}) \\ &= v(-c_{s+h} a_0^{\tau^{s+h}} - c_{s+h-1} a_1^{\tau^{s+h-1}} - \cdots - c_{h+1} a_{s-1}^{\tau^{h+1}} - c_{h-1} a_{s+1}^{\tau^{h-1}} - \cdots - c_0 a_{s+h}) \\ &\ge \operatorname{Min}\{v(c_{s+h}) + v(a_0), v(c_{s+h-1}) + v(a_1), \ldots, v(c_{h+1}) + v(a_{s-1}), \\ &\qquad v(c_{h-1}) + v(a_{s+1}), \ldots, v(c_0) + v(a_{s+h})\} \\ &\ge \operatorname{Min}\{v(c_0), v(c_1), \ldots, v(c_{h-1}), v(c_{h+1}) + 1, \ldots, v(c_{s+h}) + 1\} \end{aligned}$$

and, for $h = 0$,

$$v(c_0) \ge \operatorname{Min}\{v(c_1), v(c_2), \ldots, v(c_s)\} + 1.$$

Therefore if $v(c_{h'}) \ge q$ for all $0 \le h' \le h$ and $v(c_{h''}) \ge q - 1$ for all $h'' \ge h+1$, then $v(c_{h+1}) \ge q$. Also if $v(c_h) \ge q$ for all integers $h \ge 0$, then $v(c_0) \ge q + 1$.

Using induction on h and q, we have $v(c_h) \geq q$ for all integers $h \geq 0$ and $q \geq 1$. Hence we have $c_h = 0$ for all integers $h \geq 0$ and therefore $u = 0$.

Corollary. *Let x be as in the lemma. Suppose that $a_0 \neq 0$. If $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to Rx, then $u = 0$.*

4. Examples

Let τ be as in §3. $\sigma = \tau$ and $q = p^{f(K_0/Q_p)}$ satisfy the assumption (F) in [6, §2]. Let n and m be positive integers ($m \geq 0$ if $n = 1$) and d an integer with $0 \leq d \leq m+n-1$. Let $\Gamma_{n,m,d}$ be the n-dimensional commutative formal group over A obtained by the following special element $u_{n,m,d}$ as was done in [6]. $u_{n,m,d}$ commutes with $\mathrm{diag}(a, a, \dots, a)$ for all $a \in A_0$. Hence $\Gamma_{n,m,d}$ is an n-dimensional formal A_0-module over A. By [5, IV.21.1.4, Remarks (ii)], $\mathrm{End}_{A,A_0}(\Gamma_{n,m,d})$ coincides with the endomorphism ring $\mathrm{End}_{A,Z_p}(\Gamma_{n,m,d})$ of $\Gamma_{n,m,d}$ as a formal group over A. $u_{1,m,d} = \pi - t^{m+1}(1+t^d)$, and for $n \geq 2$,

$$u_{n,m,d} = \overbrace{\begin{pmatrix} \pi & -t & 0 & \dots & 0 \\ 0 & \pi & -t & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -t \\ -t^{m+1}(1+t^d) & 0 & 0 & \dots & \pi \end{pmatrix}}^{n} \Bigg\} n$$

We have the following generalization of [11, Theorem 2] for K.

Theorem 2. *Suppose that there exists an extension τ of τ' to an automorphism of K. Then we have*

$$\mathrm{End}_{A,A_0}(\Gamma_{n,m,d}) \cong \{\mathrm{diag}(a^{\tau^{n-1}}, a^{\tau^{n-2}}, \dots, a) | a^{\tau^d} = a^{\tau^{m+n}} = a \in A\}.$$

Therefore $\mathrm{End}_{A,A_0}(\Gamma_{n,m,d})$ is isomorphic to the valuation ring of invariants of $\tau^{(m+n,d)}$ in A.

Proof. For a ring T, T^n denotes the left free T-module of the n-dimensional row vectors over T.

Let $\{\bar{e}_i\}$ $(1 \leq i \leq n)$ be the images of the canonical basis $\{e_i\}$ $(1 \leq i \leq n)$ of A^n under the composition of the inclusion $A^n \to R^n$ and the canonical surjection $R^n \to R^n/R^n u_{n,m,d}$.

First we assume $n \geq 2$. The left R-module $R^n u_{n,m,d}$ is generated by $te_2 - \pi e_1$, $te_3 - \pi e_2, \dots, te_n - \pi e_{n-1}$ and $\pi e_n - t^{m+1}(1+t^d)e_1$. Then we have the relations $t\bar{e}_2 = \pi \bar{e}_1$, $t^2 \bar{e}_3 = \pi^{\tau+1} \bar{e}_1, \dots, t^{n-1}\bar{e}_n = \pi^{\tau^{n-2}} \pi^{\tau^{n_3}} \cdots \pi^\tau \pi \bar{e}_1$ and the annihilator of $\bar{e}_1$ is the left R-ideal

$$R(\pi^{\tau^{n-1}} \pi^{\tau^{n-2}} \cdots \pi^\tau \pi - t^{m+n}(1+t^d))$$

of R. Especially

$$R^n/R^n u_{n,m,d} \cong R/R(\pi^{\tau^{n-1}} \cdots \pi^{\tau}\pi - t^{m+n}(1+t^d))$$

is a monogenic left R-module (cf. [3, III.5.5]).

We suppose $C = (c_{ij}) \in M_n(A)$ be such that

$$f^{-1}_{n,m,d}(Cf_{n,m,d}) \in \mathrm{End}_{A,A_0}(\Gamma_{n,m,d}),$$

where $f_{n,m,d}$ is the transformer of $\Gamma_{n,m,d}$. The left $A_\tau[[t]]$-module $(A_\tau[[t]])^n u_{n,m,d} C$ is contained in $(A_\tau[[t]])^n u_{n,m,d}$ by [6, Theorem 3]. Then C gives an R-endomorphism of $R^n/R^n u_{n,m,d}$ which stabilizes $\sum_{1\le i\le n} A\bar{e}_i$. ($\sum_{1\le i\le n} A\bar{e}_i$ is a bimodule over $\mathrm{End}_{A,A_0}(\Gamma_{n,m,d})$ and A.) Therefore, we have

$$t(c_{21}\bar{e}_1 + c_{22}\bar{e}_2 + \cdots + c_{2n}\bar{e}_n) = \pi(c_{11}\bar{e}_1 + \cdots + c_{1n}\bar{e}_n),$$
$$\vdots$$
$$t(c_{n1}\bar{e}_1 + c_{n2}\bar{e}_2 + \cdots + c_{nn}\bar{e}_n) = \pi(c_{n-1\,1}\bar{e}_1 + \cdots + c_{n-1\,n}\bar{e}_n),$$
$$\pi(c_{n1}\bar{e}_1 + c_{n2}\bar{e}_2 + \cdots + c_{nn}\bar{e}_n) = t^{m+1}(1+t^d)(c_{11}\bar{e}_1 + \cdots + c_{1n}\bar{e}_n).$$

By representing $\bar{e}_i$'s with $\bar{e}_1$, we have

$$(**)\qquad \begin{cases} \{t(c_{21} + c_{22}(t^{-1}\pi) + \cdots + c_{2n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi)) \\ \qquad -\pi(c_{11} + \cdots + c_{1n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi))\}\bar{e}_1 = 0, \\ \qquad\vdots \\ \{t(c_{n1} + c_{n2}(t^{-1}\pi) + \cdots + c_{nn}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi)) \\ \qquad -\pi(c_{n-1\,1} + \cdots + c_{n-1}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi))\}\bar{e}_1 = 0, \end{cases}$$

and

$$(***)\qquad \{\pi(c_{n1} + c_{n2}(t^{-1}\pi) + \cdots + c_{nn}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi)) - t^{m+1}(1+t^d)(c_{11} + \cdots + c_{1n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi))\}\bar{e}_1 = 0.$$

We multiply $(**)$ by t^{n-1} from the left.

Since $n \le m+n-1$, by the corollary of the lemma we have $c_{i1} = 0$ $(2 \le i \le n)$, $c_{ik} = c^{\tau}_{i+1\,k+1}$ $(1 \le i,\ k \le n-1)$, and $c_{in} = 0$ $(1 \le i \le n-1)$. Hence we have $c_{ij} = 0$ if $i \ne j$ and $c_{ii} = c_{nn}^{\tau^{n-i}}$ for $1 \le i \le n-1$ and so

$$C = \mathrm{diag}(c_{nn}^{\tau^{n-1}}, c_{nn}^{\tau^{n-2}}, \ldots, c_{nn}).$$

Since the annihilator of $\bar{e}_1$ is $R(\pi^{\tau^{n-1}} \cdots \pi^{\tau}\pi - t^{m+n}(1+t^d))$, from $(***)$ we have

$$\{c_{nn}^{\tau^{-(m+1)}}(1+t^d) - t^{m+1}(1+t^d)c_{11}\}\bar{e}_1 = 0.$$

Then, by dividing the above equation by t^{m+1}, we have

$$\{c_{nn}^{\tau^{-(m+1)}} + c_{nn}^{\tau^{-(m+1)}} t^d - c_{nn}^{\tau^{n-1}} - c_{nn}^{\tau^{n+d-1}} t^d\}\overline{e}_1 = 0.$$

From $0 \leq d \leq m+n-1$, by the corollary of the lemma we have $c_{nn}^{\tau^{m+n}} = c_{nn}^{\tau^d} = c_{nn}$.

Conversely if $C = (c_{ij})$ satisfies the above conditions, then $u_{n,m,d}C = Cu_{n,m,d}$ and so $f_{n,m,d}^{-1}(Cf_{n,m,d}) \in \mathrm{End}_{A,A_0}(\Gamma_{n,m,d})$.

Hence $\mathrm{End}_{A,A_0}(\Gamma_{n,m,d})$ is isomorphic to the invariants of $\tau^{(m+n,d)}$ in A.

Finally, for $n = 1$, the analogous argument holds since the annihilator of $\overline{e}_1$ is $R(\pi - t^{m+1}(1+t^d))$.

Remark 3. (i) The field consisting of the invariants of $\tau^{(m+n,d)}$ in K has been determined more explicitly in [11, Theorem 3].

(ii) Suppose that $e(K/K_0) = 1$, $(n, m) = 1$, and k is algebraically closed for simplicity. By [3, III.5.2, Proof of Theorem 2], we have

$$R/R(\pi^{\tau^{n-1}} \cdots \pi^{\tau}\pi - t^{m+n}(1+t^d)) \cong R/R(\pi_0^n - t^{m+n}),$$

where π_0 is a prime element of K_0. Hence $\Gamma_{n,m,d} \otimes_A k$ is absolutely simple up to isogeny (cf. Proof of the corollary below).

The following corollary is a higher-dimensional analogue of [1, Theorem 5.2.2] (or [5, IV.23.2.16]).

Corollary. *For any positive integers n and h with $h \geq n+1$ ($h \geq 1$ if $n = 1$) and for any positive divisor g of h, there exists an n-dimensional formal A_0-module over A_0 of A_0-height h whose absolute A_0-endomorphism ring is the valuation ring of the unramified extension of K_0 of degree g.*

Proof. Let $K = K_0$ and $m = h - n$. Put $d = g$ if $g < h$ and $d = 0$ if $g = h$. Let K_0^* be the completion of the maximal unramified extension of K_0 and A_0^* the valuation ring of K_0^*. As in the proof of the corollary in §2, we have

$$\mathrm{END}_{*,A_0}(\Gamma_{n,m,d}) \subset \mathrm{End}_{A_0^*,A_0}(\Gamma_{n,m,d} \otimes_{A_0} A_0^*).$$

We apply Theorem 2 to $\Gamma_{n,m,d} \otimes_{A_0} A_0^*$. Thus $\mathrm{END}_{*,A_0}(\Gamma_{n,m,d})$ is contained in

$$\{\mathrm{diag}(a^{\tau^{n-1}}, a^{\tau^{n-2}}, \dots, a) | a^{\tau^g} = a \in A_0^*\}.$$

The invariants of τ^g in A_0^* coincide with the valuation ring of the unramified extension of degree g over K_0. Especially, any A_0-endomorphism of $\Gamma_{n,m,d} \otimes_{A_0} A_0^*$ is defined over the valuation ring of the finite extension of K_0. Hence the converse inclusion follows.

For the functor M in [2], we have

$$M(\Gamma_{n,m,d} \otimes_{A_0} k_0) \cong A_{0\tau}[[t]]^n/(A_{0\tau}[[t]]^n u_{n,m,d})$$

as in [4, V.2]. Thus we have

$$[k_0 \otimes_{A_0} M(\Gamma_{n,m,d} \otimes_{A_0} k_0) : k_0] = m + n$$

and therefore $\Gamma_{n,m,d}$ is of A_0-height $m + n = h$.

Remark 4. If $(n, m) = 1$, then $\Gamma_{n,m,d} \otimes_{A_0} k_0$ is absolutely simple up to isogeny as in Remark 3(ii) (cf. [3, III.5.5]).

REFERENCES

1. L. H. Cox, *Formal A-modules over* $\mathfrak{p}$*-adic integer rings*, Compositio Math. **29** (1974), 287–308.
2. J-M. Decauwert, *Classification des A-modules formels*, C. R. Acad. Sci. Paris Ser. A **282** (1976), 1413–1416.
3. J. Dieudonné, *Introduction to the theory of formal groups*, Dekker, New York, 1973.
4. J-M. Fontaine, *Groupes p-divisibles sur les corps locaux*, Astérisques **47–48** (1977), 1–262.
5. M. Hazewinkel, *Formal groups and applications*, Academic Press, New York, 1978.
6. T. Honda, *On the theory of commutative formal groups*, J. Math. Soc. Japan **22** (1970), 213–246.
7. J. Lubin, *One-parameter formal Lie groups over* $\mathfrak{p}$*-adic integer rings*, Ann. of Math. **80** (1964), 464–484.
8. Y. I. Manin, *The theory of commutative formal groups over fields of finite characteristic*, Russian Math. Surveys **18** (1963), 1–83.
9. T. Nakamura, *On two-dimensional formal groups over the prime field of characteristic* $p > 0$, J. Algebra **88** (1984), 228–237.
10. W. C. Waterhouse, *On p-divisible groups over complete valuation rings*, Ann. of Math. **95** (1972), 55-65.
11. Y. Yamasaki, *On the endomorphism rings of Honda groups* $H_{n,m}$ *over* $\mathfrak{p}$*-adic integer rings*, Osaka J. Math. **12** (1975), 457–472.

DEPARTMENT OF MATHEMATICAL SCIENCES, COLLEGE OF SCIENCE AND ENGINEERING, TOKYO DENKI UNIVERSITY, HATOYAMA-MACHI, HIKI-GUM, SAITAMA-KEN, 350-03, JAPAN